Existence of stochastic entropy solutions for stochastic scalar balance laws with Lipschitz vector fields*

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Abstract In this paper, we consider a scalar stochastic balance law and gain the existence for stochastic entropy solutions. Our proof relies on the BGK approximation and the generalized Itô formula. Moreover, as an application, we derive the existence of stochastic entropy solutions for stochastic Buckley-Leverett type equations.

Keywords: Existence; Stochastic entropy solution; BGK approximation; Itô formula

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1 Introduction

In this paper, we study the first-order scalar balance law with Stratonovich type perturbations:

$$\partial_t \rho(t,x) + \mathrm{div}_x(B(\rho)) + \partial_{x_i}\rho(t,x) \circ \dot{M}_i(t) = A(t,\rho), \quad \text{in } \Omega \times (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

where $M_i(t) = \int_0^t \sigma_{i,j}(s) dW_j(s)$, $(1 \leqslant i,j \leqslant d)$ and $\sigma_{i,j} \in L^2_{loc}([0,\infty))$. The flux function B is assumed to be of class $\mathcal{C}^1$: $B \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d)$ and the forcing A is supposed to be Lipschitz: $A \in L^1_{loc}([0,\infty); W^{1,\infty}(\mathbb{R}))$ and $A(t,0) = 0$. To formulate the Cauchy problem, we presume

$$\rho(t,x)|_{t=0} = \rho_0(x), \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

here ρ_0 is a non-random function.

The Cauchy problem for (1.1), (1.2) in the case of $\sigma_{i,j} = 0$ has been studied in Kružkov [1]. He gains the uniqueness of entropy solutions as well as the existence.

In the case of the perturbations are quasi-linear dependent, i.e.

$$\partial_t \rho(t,x) + \partial_{x_i} B_i(\rho) \circ \dot{W}_i(t) = A(t,\rho), \quad \text{in } \Omega \times (0,\infty) \times \mathbb{R}^d, \tag{1.3}$$

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Lions, Perthame and Souganidis [2] develop a pathwise theory for weak $L^1 \cap L^\infty$-solutions for which A vanishes and $B \in \mathcal{C}^2(\mathbb{R};\mathbb{R}^d)$.

When the stochastic perturbations in (1.1) is replaced by a multiplicative noise, which in our present setting takes the form

$$\partial_t \rho(t,x) + \mathrm{div}_x(B(\rho)) = A(t,\rho)\dot{W}(t), \quad \text{in } \ \Omega \times (0,\infty) \times \mathbb{R}^d, \tag{1.4}$$

it has been studied by many other researchers [3-5]. For example, in [3], Chen, Ding and Karlsen concern with $(1.4), (1.1)$ with $A(t,\rho) = A(\rho)$, then they supply an existence theory of stochastic entropy solutions for $\mathcal{C}^2$ flux.

In above papers, existence of (stochastic) entropy solutions are proved via approximation by (stochastic) parabolic equations. Using a differential philosophy, Hofmanová in [6] proceeds a stochastic BGK approximation, and the existence of stochastic entropy solution is proved for $\mathcal{C}^{4,\alpha}$ $(\alpha > 0)$ flux and Lipschitz forcing.

All above mentioned works for stochastic balance laws are concentrated on $\mathcal{C}^2$ (or $\mathcal{C}^{4,\alpha}$) flux, there are relatively few research works for well-posedness of $(1.1), (1.2)$ for $\mathcal{C}^1$ flux. Our purpose now is to raise a well-posedness theory for $L^1 \cap L^\infty$-solutions to (1.1).(1.2) on stochastic entropy solutions to the level of Kružkov theory. Our present work is a fellow-up of Wei and Liu's work [7]. In [7], the authors put forward the following result:

Lemma 1.1 ***(Stochastic kinetic formula)*** *Assume that* $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. *(i) Let* ρ *be a stochastic entropy solution of* $(1.1), (1.2)$ *and set*

$$u(t,x,v) = \chi_{\rho(t,x)}(v) = \begin{cases} 1, & \textit{when } 0 < v < \rho, \\ -1, & \textit{when } \rho < v < 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{1.5}$$

then

$$u \in L^\infty(\Omega; L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d_x; L^1(\mathbb{R}_v)))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d_x \times \mathbb{R}_v \times \Omega)), \tag{1.6}$$

and it is a stochastic weak solution of the linear stochastic transport problem

$$\begin{cases} \partial_t u + b(v)\cdot \nabla_x u + \partial_{x_i} u \circ \dot{M}_i(t) + A(t,v)\partial_v u = \partial_v m, & \textit{in } \ \Omega \times (0,\infty) \times \mathbb{R}^d_x \times \mathbb{R}_v, \\ u(t,x,v)|_{t=0} = \chi_{\rho_0}(v), & \textit{in } \ \mathbb{R}^d_x \times \mathbb{R}_v, \end{cases} \tag{1.7}$$

where $b = B'$, $0 \leqslant m \in L^1(\Omega; \mathcal{D}'([0,\infty) \times \mathbb{R}^d_x \times \mathbb{R}_v))$, *satisfying, for any* $T > 0$ *and for almost all* $\omega \in \Omega$, m *is bounded on* $[0,T] \times \mathbb{R}^d_x \times \mathbb{R}_v$, *supported in* $[0,T] \times \mathbb{R}^d_x \times [-K,K]$ $(K = \|\rho\|_{L^\infty((0,T)\times\mathbb{R}^d\times\Omega)})$, *continuous in* t, *here the continuous is interpreted*

$$m([0,s] \times \mathbb{R}^d_x \times \mathbb{R}_v) \to m([0,t] \times \mathbb{R}^d_x \times \mathbb{R}_v), \quad \textit{as } \ s \to t. \tag{1.8}$$

(ii) Let $u \in L^\infty(\Omega; L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d_x; L^1(\mathbb{R}_v)))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d_x \times \mathbb{R}_v \times \Omega))$ *be a stochastic weak solution of* (1.7) *(i.e.* u *yields* (1.7) *in the sense of distributions), with* m *meeting the properties stated in* (i). *Then*

$$\rho(t,x) = \int_{\mathbb{R}} u(t,x,v)dv \in L^\infty(\Omega; L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d \times \Omega)), \tag{1.9}$$

and it is a stochastic entropy solution of $(1.1), (1.2)$.

Then they derive the uniqueness of stochastic entropy solutions for (1.1), (1.2). Whence, to establish the well-posedness for (1.1), (1.2), it remains to found the existence of solutions, and it is the main interest of the paper here.

This paper is organized as follows. In Section 2, we review some notions. Section 3 is devoted to the existence on stochastic entropy solutions for (1.1), (1.2).

As usual, the notation used here is mostly standard. $\mathcal{D}(\mathbb{R}^d)$ is the space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with compact support. Correspondingly, $\mathcal{D}_+(\mathbb{R}^d)$ represents the non-negative elements in $\mathcal{D}(\mathbb{R}^d)$. $\mathbb{N}$ denotes the set consisting of all natural numbers. $W(t) = (W_1(t), W_2(t), \dots, W_d(t))^\top$ stands for the standard d-dimensional Wiener procss on the classical Wiener space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geqslant 0})$, i.e. Ω is the space of all continuous functions from $[0,\infty)$ to $\mathbb{R}^d$ with locally uniform convergence topology, $\mathcal{F}$ is the Borel σ-field, P is the Wiener measure, $(\mathcal{F}_t)_{t\geqslant 0}$ is the natural filtration generated by the coordinate process $W_t(\omega) = \omega_t$. The stochastic integration with a notation $\circ$ is interpreted in Stratonovich sense and the others is Itô's. The $C(T)$ denotes a positive constant depends only on T, whose values may change in different places. For a given measurable function f, f^+ is its positive portion, defined by $1_{f\geqslant 0} f$, and $f^- = [-f]^+$. The summation convention is enforced throughout this article, wherein summation is understood with respect to repeated indices.

2 Stochastic entropy solutions

In this short section, we introduce some notions and review the stochastic kinetic formula for future use.

Definition 2.1 *Let $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. $\rho \in L^\infty(\Omega; L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d \times \Omega))$ is a stochastic weak solution of* (1.1), (1.2), *if the random field ρ is $\mathcal{F}_t$-adapted, and for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, with probability one, the below identity*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x)\rho(t,x)dx - \int_0^t \int_{\mathbb{R}^d} B(\rho)\cdot\nabla_x\varphi(x)dxds - \int_0^t M_i(\circ ds)\int_{\mathbb{R}^d}\partial_{x_i}\varphi(x)\rho(s,x)dx \\
= \; &\int_{\mathbb{R}^d}\varphi(x)\rho_0(x)dx + \int_0^t\int_{\mathbb{R}^d} A(s,\rho)\varphi(x)dxds, \qquad (2.1)
\end{aligned}$$

holds true, for all $t \in [0,\infty)$. The stochastic weak solution is call a stochastic entropy solution, if for any $\eta(\rho) \in \Xi$,

$$\partial_t\eta(\rho) + div(Q(\rho)) + \partial_{x_i}\eta(\rho)\circ \dot{M}_i(t) \leqslant h(t,\rho), \;\; P-a.s. \; \omega\in\Omega, \qquad (2.2)$$

in the sense of distributions, where

$$Q(\rho) = \int^\rho \eta'(v)b(v)dv, \;\; h(t,\rho) = A(t,\rho)\eta'(\rho), \qquad (2.3)$$

and

$$\Xi = \{c_0\rho + \sum_{k=1}^n c_k|\rho-\rho_k|, \quad c_0,\rho_k,c_k\in\mathbb{R} \;\; \textit{are constants}\}, \quad n\in\mathbb{N}.$$

3 Existence of stochastic entropy solutions

In this section, we intend to found the fundamental existence results on stochastic entropy solutions to (1.1), (1.2). Inspiring by Lemma 2.1 stated in introduction, it suffices to establish the existence on stochastic weak solutions for (1.7). Initially, we give a remark.

Remark 3.1 *m is continuous in t (see (1.8)), so the $\mathcal{F}_t$-adapted random field u which lies in $L^\infty(\Omega; L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d_x; L^1(\mathbb{R}_v)))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d_x \times \mathbb{R}_v \times \Omega))$ is a stochastic weak solution of (1.7) if and only if, for any $\phi(x,v) \in \mathcal{D}(\mathbb{R}^d_x \times \mathbb{R}_v)$, with probability one,*

$$\begin{aligned}
&\int_{\mathbb{R}^d_x\times\mathbb{R}_v} \phi(x,v)u(t,x,v)dxdv - \int_0^t \int_{\mathbb{R}^d_x\times\mathbb{R}_v} b(v)\cdot\nabla_x\phi(x,v)dxdvds \\
= &\int_{\mathbb{R}^d_x\times\mathbb{R}_v} \phi(x,v)u_0(x,v)dxdv + \int_0^t \int_{\mathbb{R}^d_x\times\mathbb{R}_v} \partial_v[A(s,v)\phi(x,v)]u(s,x,v)dxdvds \\
&- \int_0^t \int_{\mathbb{R}^d_x\times\mathbb{R}_v} \partial_v\phi(x,v)m(dx,dv,ds) + \int_0^t M_i(\circ ds)\int_{\mathbb{R}^d_x\times\mathbb{R}_v} \partial_{x_i}\phi(x,v)u(s,x,v)dxdv
\end{aligned}$$

is legitimate, for all $t \in [0,\infty)$.

We are now in a position to give our main result.

Theorem 3.1 ***(Existence)*** *Let B, σ and A yield the conditions stated in introduction. If $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then there exists a stochastic entropy solution of the Cauchy problem (1.1), (1.2).*

Proof. The conclusion will be reached in three steps.

• **Step 1:** $\sigma = 0$. Now (1.7) becomes to

$$\begin{cases} \partial_t u(t,x,v) + b(v)\cdot\nabla_x u(t,x,v) + A(t,v)\partial_v u(t,x,v) = \partial_v m, & \text{in } (0,\infty)\times\mathbb{R}^d_x\times\mathbb{R}_v, \\ u(t,x,v)|_{t=0} = \chi_{\rho_0(x)}(v), & \text{in } \mathbb{R}^d_x\times\mathbb{R}_v. \end{cases} \tag{3.1}$$

We begin with building the existence of weak solutions for (3.1) by using the BGK approximation, i.e., for $\varepsilon > 0$, we regard (3.1) as the $\varepsilon \to 0$ limit of the integro-differential equation

$$\begin{cases} \partial_t u_\varepsilon(t,x,v) + b(v)\cdot\nabla_x u_\varepsilon + A(t,v)\partial_v u_\varepsilon = \frac{1}{\varepsilon}[\chi_{\rho_\varepsilon(t,x)} - u_\varepsilon], & \text{in } (0,\infty)\times\mathbb{R}^d_x\times\mathbb{R}_v, \\ u_\varepsilon(t,x,v)|_{t=0} = \chi_{\rho_0(x)}(v), & \text{in } \mathbb{R}^d_x\times\mathbb{R}_v, \end{cases} \tag{3.2}$$

where $\rho_\varepsilon(t,x) = \int_{\mathbb{R}} u_\varepsilon(t,x,v)dv$.

• **Assertion 1:** (3.2) is well-posed in $L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d_x\times\mathbb{R}_v)) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d_x\times\mathbb{R}_v))$.

Clearly, $(3.2)_1$ grants an equivalent presentation

$$\partial_t Z_\varepsilon + b(v)\cdot\nabla_x Z_\varepsilon + A(t,v)\partial_v Z_\varepsilon = \frac{1}{\varepsilon}e^{\frac{t}{\varepsilon}}\chi_{e^{-\frac{t}{\varepsilon}}\tilde{\rho}_\varepsilon}(v),$$

here

$$Z_\varepsilon(t,x,v) = e^{\frac{t}{\varepsilon}}u_\varepsilon(t,x,v),\ \tilde{\rho}_\varepsilon = \int_{\mathbb{R}} Z_\varepsilon(t,x,v)dv.$$

Due to the assumptions

$$B \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d),\ A \in L^1_{loc}([0,\infty); W^{1,\infty}(\mathbb{R})),$$

there is a unique global solution to the ODE

$$\frac{d}{dt}(X(t,x,v),V(t,v))^\top = (b(V),A(t,V))^\top, \quad \text{with} \quad (X(t,x,v),V(t,v))^\top|_{t=0} = (x,v)^\top, \tag{3.3}$$

for any $(x,v) \in \mathbb{R}_x^d \times \mathbb{R}_v$.

Therefore, along the direction (3.3),

$$Z_\varepsilon(t,X(t),V(t)) = \frac{1}{\varepsilon}\int_0^t e^{\frac{s}{\varepsilon}}\chi_{e^{-\frac{s}{\varepsilon}}\tilde{\rho}_\varepsilon(s,X(s,x,v))}(V(s,v))ds + \chi_{\rho_0(x)}(v),$$

i.e.

$$u_\varepsilon(t,X(t),V(t)) = \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}\chi_{\rho_\varepsilon(s,X(s,x,v))}(V(s,v))ds + e^{-\frac{t}{\varepsilon}}\chi_{\rho_0(x)}(v).$$

Define $J(t,V) = |\partial_v V(t,v)|$, thanks to Euler's formula, then

$$\exp(-\int_0^t [\partial_v A(s,V(s))]^- ds) \leqslant J(t,V) \leqslant \exp(\int_0^t [\partial_v A(s,V(s))]^+ ds), \tag{3.4}$$

whence the inverse of the mapping $(x,v)^\top \mapsto (X,V)^\top$ exists and it forms a flow of homeomorphic. We thus have

$$u_\varepsilon(t,x,v) = \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}\chi_{\rho_\varepsilon(s,X_{t,s}(x,v))}(V_{t,s}(v))ds + e^{-\frac{t}{\varepsilon}}\chi_{\rho_0(X_{t,0}(x,v))}(V_{t,0}(v)), \tag{3.5}$$

where $(X_{t,s}(x,v),V_{t,s}(v))^\top = [(X_{s,t}^{-1}(x,v),V_{s,t}^{-1}(v))^\top]$, i.e.

$$\begin{cases} \frac{d}{dt}(X_{s,t}(x,v),V_{s,t}(v))^\top = (b(V_{s,t}),A(t,V_{s,t}))^\top, \ t \geqslant s, \\ (X_{s,t}(x,v),V_{s,t}(v))^\top|_{t=s} = (X(s,x,v),V(s,v))^\top, \end{cases}$$

and $(X_{s,t}(x,v),V_{s,s}(v))^\top = (X(t,X(s,x,v),V(s,v)),V(t,V(s,v)))^\top$.

For any $u \in L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}_x^d\times\mathbb{R}_v))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}_x^d\times\mathbb{R}_v))$, we define a mapping S_ε by:

$$(S_\varepsilon u)(t,x,v) = \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}\chi_{\rho^u(s,X_{t,s}(x,v))}(V_{t,s}(v))ds + e^{-\frac{t}{\varepsilon}}\chi_{\rho_0^u(X_{t,0}(x,v))}(V_{t,0}(v)), \tag{3.6}$$

here

$$\rho^u(t,x) = \int_{\mathbb{R}} u(t,x,v)dv, \quad \rho_0^u(x) = \int_{\mathbb{R}} u(0,x,v)dv = \rho_0(x).$$

We claim that S_ε is well-defined in $L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}_x^d\times\mathbb{R}_v))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}_x^d\times\mathbb{R}_v))$ and locally (in time) contractive in $\mathcal{C}([0,\infty);L^1(\mathbb{R}_x^d\times\mathbb{R}_v))$.

Initially, we collate that (3.6) is well-defined. Indeed,

$$\|S_\varepsilon u\|_{L^\infty([0,T]\times\mathbb{R}_x^d\times\mathbb{R}_v)} \leqslant 1, \tag{3.7}$$

and for any $0 < T < \infty$,

$$\sup_{0\leqslant t\leqslant T}\Big|\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}ds\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\chi_{\rho^u(s,X_{t,s}(x,v))}(V_{t,s}(v))dxdv + e^{-\frac{t}{\varepsilon}}\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\chi_{\rho_0^u(X_{t,0}(x,v))}(V_{t,0}(v))dxdv\Big|$$

$$
\begin{aligned}
&= \sup_{0\leqslant t\leqslant T}\Big|\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}ds\int_{\mathbb{R}^d_x\times\mathbb{R}_v}\chi_{\rho^u(s,x)}(v)\exp(\int_s^t\partial_v A(r,V_{s,r}(v))dr)dxdv\\
&\quad + e^{-\frac{t}{\varepsilon}}\int_{\mathbb{R}^d_x\times\mathbb{R}_v}\chi_{\rho^u_0(x)}(v)\exp(\int_0^t\partial_v A(r,V_{0,r}(v))dr)dxdv\Big|\\
&\leqslant \exp(\int_0^T\|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(t)dt)\Big[(1-e^{-\frac{T}{\varepsilon}})\|u\|_{\mathcal{C}([0,T];L^1(\mathbb{R}^d_x\times\mathbb{R}_v))}+\|\rho^u_0\|_{L^1(\mathbb{R}^d_x\times\mathbb{R}_v)}\Big],
\end{aligned}
\tag{3.8}
$$

thus (3.6) is meaningful.

For any $f,g\in L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}^d_x\times\mathbb{R}_v))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}^d_x\times\mathbb{R}_v))$, an analogue calculation of (3.8) also leads to

$$
\begin{aligned}
&\|S_\varepsilon f-S_\varepsilon g\|_{\mathcal{C}([0,T];L^1(\mathbb{R}^d_x\times\mathbb{R}_v))}\\
&\leqslant \sup_{0\leqslant t\leqslant T}\Big|\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}ds\int_{\mathbb{R}^d_x\times\mathbb{R}_v}|\chi_{\rho^f(s,X_{t,s}(x,v))}(V_{t,s}(v))-\chi_{\rho^g(s,X_{t,s}(x,v))}(V_{t,s}(v))|dxdv\\
&\quad + e^{-\frac{t}{\varepsilon}}\int_{\mathbb{R}^d_x\times\mathbb{R}_v}|\chi_{\rho^f_0(X_{t,0}(x,v))}(V_{t,0}(v))-\chi_{\rho^g_0(X_{t,0}(x,v))}(V_{t,0}(v))|dxdv\Big|\\
&= \sup_{0\leqslant t\leqslant T}\Big|\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}ds\int_{\mathbb{R}^d_x\times\mathbb{R}_v}|\chi_{\rho^f(s,x)}(v)-\chi_{\rho^g(s,x)}(v)|\exp(\int_s^t\partial_v A(r,V_{s,r})dr)dxdv\\
&\quad + e^{-\frac{t}{\varepsilon}}\int_{\mathbb{R}^d_x\times\mathbb{R}_v}|\chi_{\rho^f_0(x)}(v)-\chi_{\rho^g_0(x)}(v)|\exp(\int_0^t\partial_v A(r,V_{0,r}(v))dr)dxdv\Big|\\
&\leqslant \exp(\int_0^T\|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}dt)\Big[(1-e^{-\frac{T}{\varepsilon}})\|f-g\|_{\mathcal{C}([0,T];L^1(\mathbb{R}^d_x\times\mathbb{R}_v))}+\|f_0-g_0\|_{L^1(\mathbb{R}^d_x\times\mathbb{R}_v)}\Big].
\end{aligned}
\tag{3.9}
$$

In particular, if $f_0=g_0=\chi_{\rho_0}$, from (3.9),

$$
\|S_\varepsilon f-S_\varepsilon g\|_{\mathcal{C}([0,T];L^1(\mathbb{R}^d_x\times\mathbb{R}_v))}\leqslant\exp(\int_0^T\|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(t)dt)(1-e^{-\frac{T}{\varepsilon}})\|f-g\|_{\mathcal{C}([0,T];L^1(\mathbb{R}^d_x\times\mathbb{R}_v))},
$$

for any $T>0$.

Given above $T>0$ we select $T_1>0$ so small that $\exp(\int_0^T\|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(t)dt)(1-e^{-\frac{T_1}{\varepsilon}})<1$. Then we apply the Banach fixed point theorem to find a unique $u_\varepsilon\in\mathcal{C}([0,T_1];L^1(\mathbb{R}^d_x\times\mathbb{R}_v))$ solving the Cauchy problem (3.2). By (3.7), $u_\varepsilon\in L^\infty([0,T];L^\infty(\mathbb{R}^d_x\times\mathbb{R}_v))$, so $u_\varepsilon(T_1)\in L^1(\mathbb{R}^d_x\times\mathbb{R}_v))\cap L^\infty(\mathbb{R}^d_x\times\mathbb{R}_v)$. We then repeat the argument above to extend our solution to the time interval $[T_1,2T_1]$. Continuing, after finitely many steps we construct a solution existing on the interval $(0,T)$ for any $T>0$. From this, we demonstrate that there exists a unique $u_\varepsilon\in\mathcal{C}([0,\infty);L^1(\mathbb{R}^d_x\times\mathbb{R}_v))\cap L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}^d_x\times\mathbb{R}_v))$ solving the Cauchy problem (3.2).

• **Assertion 2: (Comparison principle)**. For any $\rho_0,\tilde{\rho_0}\in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$, the allied solutions u_ε and $\tilde{u}_\varepsilon$ of (3.2) satisfy

$$
\begin{aligned}
\|[u_\varepsilon(t)-\tilde{u}_\varepsilon(t)]^+\|_{L^1(\mathbb{R}^d_x\times\mathbb{R}_v)} &\leqslant \exp(\int_0^t\|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds)\|[\chi_{\rho_0}-\chi_{\tilde{\rho}_0}]^+\|_{L^1(\mathbb{R}^d_x\times\mathbb{R}_v)}\\
&= \exp(\int_0^t\|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds)\|[\rho_0-\tilde{\rho}_0]^+\|_{L^1(\mathbb{R}^d)},
\end{aligned}
\tag{3.10}
$$

$$\|\rho_\varepsilon(t)-\tilde{\rho}_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \leqslant \exp\Big(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds\Big)\|\rho_0-\tilde{\rho}_0\|_{L^1(\mathbb{R}^d)}, \tag{3.11}$$

$$\|\rho_\varepsilon(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \exp\Big(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds\Big)\|\rho_0\|_{L^\infty(\mathbb{R}^d)}. \tag{3.12}$$

Furthermore, if $\rho_0 \leqslant \tilde{\rho}_0$, for for almost all $(t,x,v) \in (0,\infty)\times\mathbb{R}^d_x\times\mathbb{R}_v$, and almost all $(t,x) \in (0,\infty)\times\mathbb{R}^d$,

$$u_\varepsilon(t,x,v) \leqslant \tilde{u}_\varepsilon(t,x,v), \quad \rho_\varepsilon(t,x) \leqslant \tilde{\rho}_\varepsilon(t,x). \tag{3.13}$$

(3.13) holds mutatis mutandis from (3.10) and (1.5), it is sufficient to show (3.10) – (3.12). Since the calculation for (3.11) and (3.12) is analogue of (3.10), we only show (3.10) here. Let $\lambda_\varepsilon = [u_\varepsilon - \tilde{u}_\varepsilon]^+$, by a tedious approximation argument, it meets

$$\partial_t\lambda_\varepsilon(t,x,v) + b(v)\cdot\nabla_x\lambda_\varepsilon + A(t,v)\partial_v\lambda_\varepsilon = \frac{1}{\varepsilon}[\chi_{\rho_\varepsilon(t,x)} - \chi_{\tilde{\rho}_\varepsilon(t,x)} - (u_\varepsilon-\tilde{u}_\varepsilon)]\mathrm{sign}\lambda_\varepsilon, \tag{3.14}$$

in $(0,\infty)\times\mathbb{R}^d_x\times\mathbb{R}_v$, with the initial data

$$\lambda_\varepsilon|_{t=0} = [\chi_{\rho_0(x)}(v) - \chi_{\tilde{\rho}_0(x)}(v)]^+, \quad \text{in } \mathbb{R}^d_x\times\mathbb{R}_v. \tag{3.15}$$

Notice that,

$$[\chi_{\rho_\varepsilon(t,x)} - \chi_{\tilde{\rho}_\varepsilon(t,x)} - (u_\varepsilon-\tilde{u}_\varepsilon)]\mathrm{sign}\lambda_\varepsilon = [\chi_{\rho_\varepsilon(t,x)} - \chi_{\tilde{\rho}_\varepsilon(t,x)}]\mathrm{sign}\lambda_\varepsilon - \lambda_\varepsilon, \tag{3.16}$$

and

$$\int_{\mathbb{R}} [\chi_{\rho_\varepsilon(t,x)}(v) - \chi_{\tilde{\rho}_\varepsilon(t,x)}(v)]\mathrm{sign}\lambda_\varepsilon(t,x,v)dv \leqslant \int_{\mathbb{R}} \lambda_\varepsilon(t,x,v)dv. \tag{3.17}$$

Indeed, when $\rho_\varepsilon \leqslant \tilde{\rho}_\varepsilon$, (3.17) is nature and reversely,

$$\begin{aligned}\int_{\mathbb{R}} [\chi_{\rho_\varepsilon(t,x)}(v) - \chi_{\tilde{\rho}_\varepsilon(t,x)}(v)]\mathrm{sign}\lambda_\varepsilon(t,x,v)dv &\leqslant \int_{\mathbb{R}} [\chi_{\rho_\varepsilon(t,x)}(v) - \chi_{\tilde{\rho}_\varepsilon(t,x)}(v)]dv \\ &= \int_{\mathbb{R}} [u_\varepsilon - \tilde{u}_\varepsilon]dv \\ &\leqslant \int_{\mathbb{R}} \lambda_\varepsilon(t,x,v)dv.\end{aligned}$$

By (3.16), (3.17), from (3.14) it follows that

$$\partial_t \int_{\mathbb{R}} \lambda_\varepsilon(t,x,v)dv + \int_{\mathbb{R}} b(v)\cdot\nabla_x\lambda_\varepsilon dv \leqslant \int_{\mathbb{R}} \partial_v A(t,v)\lambda_\varepsilon dv \leqslant \|[\partial_v A(t)]^+\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\lambda_\varepsilon dv,$$

which suggests that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\frac{d}{dt}\int_{\mathbb{R}^d_x\times\mathbb{R}_v} \lambda_\varepsilon(t,x,v)\varphi(x)dxdv \leqslant \int_{\mathbb{R}^d_x\times\mathbb{R}_v} b(v)\cdot\nabla_x\varphi\lambda_\varepsilon dxdv + \|[\partial_v A(t)]^+\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}^d_x\times\mathbb{R}_v}\lambda_\varepsilon\varphi dxdv.$$

For any $k \in \mathbb{N}$, we can choose φ such that for any $0 \leqslant |x| \leqslant k$, $\varphi(x) = 1$, then by letting k tend to infinity, one deduces

$$\frac{d}{dt}\int_{\mathbb{R}_x^d\times\mathbb{R}_v} \lambda_\varepsilon(t,x,v)dxdv \leqslant \|[\partial_v A(t)]^+\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}_x^d\times\mathbb{R}_v} \lambda_\varepsilon(t,x,v)dxdv. \tag{3.18}$$

On account of the fact: for any $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\int_{\mathbb{R}} [\chi_{\alpha_1}(v) - \chi_{\alpha_2}(v)]^+ dv = [\alpha_1 - \alpha_2]^+, \tag{3.19}$$

from (3.18), by (3.15) and a Grönwall type argument, one arrives (3.10)

• **Assertion 3:** With locally uniform convergence topology, $\{u_\varepsilon\}$ is pre-compact in $\mathcal{C}([0,\infty); L^1(\mathbb{R}_x^d\times\mathbb{R}_v))$ and $\{\rho_\varepsilon\}$ is pre-compact in $\mathcal{C}([0,\infty); L^1(\mathbb{R}^d))$.

From (3.10) (with a slight change), we have for any $(\tilde{x},\tilde{v}) \in \mathbb{R}_x^d\times\mathbb{R}_v$, $t \in (0,\infty)$,

$$\begin{aligned}
&\|u_\varepsilon(t,\tilde{x}+\cdot,\tilde{v}+\cdot) - u_\varepsilon(t,\cdot,\cdot)\|_{L^1(\mathbb{R}_x^d\times\mathbb{R}_v))}\\
\leqslant\ & \frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}\|u_\varepsilon(s,\tilde{x}+\cdot,\tilde{v}+\cdot) - u_\varepsilon(s,\cdot,\cdot)\|_{L^1(\mathbb{R}_x^d\times\mathbb{R}_v)}\exp(\int_s^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(r)dr)ds\\
&+e^{-\frac{t}{\varepsilon}}\int_{\mathbb{R}_x^d\times\mathbb{R}_v} |\chi_{\rho_0(x+\tilde{x})}(v+\tilde{v}) - \chi_{\rho_0(x)}(v)|dxdv\exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(r)dr).
\end{aligned}$$

Thus

$$\begin{aligned}
&\|u_\varepsilon(t,\tilde{x}+\cdot,\tilde{v}+\cdot) - u_\varepsilon(t,\cdot,\cdot)\|_{L^1(\mathbb{R}_x^d\times\mathbb{R}_v)}\\
\leqslant\ & \int_{\mathbb{R}_x^d\times\mathbb{R}_v} |\chi_{\rho_0(x+\tilde{x})}(v+\tilde{v}) - \chi_{\rho_0(x)}(v)|dxdv\exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds).
\end{aligned}$$

With the aid of (3.19), then for $\tilde{v} = 0$, it follows that

$$\begin{aligned}
&\|\rho_\varepsilon(t,\tilde{x}+\cdot) - \rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^d)}\\
=\ & \int_{\mathbb{R}_x^d}\Big|\int_{\mathbb{R}_v} u_\varepsilon(t,\tilde{x}+x,v)dv - \int_{\mathbb{R}} u_\varepsilon(t,x,v)dv\Big|dx\\
\leqslant\ & \int_{\mathbb{R}_x^d\times\mathbb{R}_v} |u_\varepsilon(t,\tilde{x}+x,v) - u_\varepsilon(t,x,v)|dxdv\\
\leqslant\ & \int_{\mathbb{R}_x^d\times\mathbb{R}_v} |\chi_{\rho_0(x+\tilde{x})}(v) - \chi_{\rho_0(x)}(v)|dxdv\exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(r)dr),
\end{aligned}$$

which implies for any $0 < T < \infty$, $\{u_\varepsilon\}$ is contained in a compact set of $\mathcal{C}([0,T]; L^1_{loc}(\mathbb{R}_x^d\times\mathbb{R}_v))$, $\{\rho_\varepsilon\}$ is pre-compact in $\mathcal{C}([0,T]; L^1_{loc}(\mathbb{R}^d))$. Hence by appealing to the Arzela-Ascoli theorem, with any sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$ as $k\to\infty$, is associated two subsequences (for ease of notation, we also denote them by themselves) $\{u_{\varepsilon_k}\}$ and $\{\rho_{\varepsilon_k}\}$, such that

$$u_{\varepsilon_k} \longrightarrow u \in \mathcal{C}([0,T]; L^1_{loc}(\mathbb{R}_x^d\times\mathbb{R}_v)), \quad \rho_{\varepsilon_k} \longrightarrow \rho \in \mathcal{C}([0,T]; L^1_{loc}(\mathbb{R}^d)), \quad \text{as} \quad k\to\infty.$$

On the other hand, by (3.7) and the lower semi-continuity,

$$u \in L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}_x^d\times\mathbb{R}_v)) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}_x^d\times\mathbb{R}_v)),$$

$$\rho \in L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^d)) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}^d)).$$

- **Assertion 4:** $\frac{1}{\varepsilon}[\chi_{\rho_\varepsilon} - u_\varepsilon] = \partial_v m_\varepsilon$, where $m_\varepsilon \geqslant 0$ is continuous in t and bounded uniformly in ε.

Let $(t,x) \in (0,\infty) \times \mathbb{R}^d$ be fixed, assuming without loss of generality that $\rho_\varepsilon \geqslant 0$, define

$$m_\varepsilon(t,x,v) = \frac{1}{\varepsilon}\int_{-\infty}^{v} [\chi_{\rho_\varepsilon(t,x)}(r) - u_\varepsilon(t,x,r)]dr.$$

In view of (3.5),

$$u_\varepsilon(t,x,r) \in \left\{ \begin{array}{ll} [0,1], & \text{when } r > 0, \\ [-1,0], & \text{when } r < 0. \end{array} \right.$$

Hence $m_\varepsilon(t,x,v)$ is nondecreasing on $(-\infty,\rho_\varepsilon)$ and nonincreasing on $[\rho_\varepsilon,\infty)$. On the other side, $m_\varepsilon(t,x,-\infty) = m_\varepsilon(t,x,\infty) = 0$, we conclude $m_\varepsilon \geqslant 0$.

Since $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, owing to (3.4), (3.5), (3.12) and condition $A \in L^1_{loc}([0,\infty); W^{1,\infty}(\mathbb{R}))$,

$$\mathrm{supp} m_\varepsilon \subset [0,T] \times \mathbb{R}^d_x \times [-K,K],$$

where $K = \|\rho\|_{L^\infty((0,T)\times\mathbb{R}^d)} \exp(\int_0^T \|\partial_v A(t)\|_{L^\infty(\mathbb{R})} ds)$.

For above fixed $T > 0$,

$$\begin{aligned} & \int_0^T dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}} m_\varepsilon(t,x,v) dv \\ = & \int_0^T dt \int_{\mathbb{R}^d} dx \int_{-K}^{K} dv \int_{-K}^{v} [\partial_t u_\varepsilon + b(r)\cdot \nabla_x u_\varepsilon + A(t,r)\partial_r u_\varepsilon] dr \\ = & \int_0^T dt \int_{\mathbb{R}^d} dx \int_{-K}^{K} dv \int_{-K}^{v} [\partial_t u_\varepsilon(t,x,r) + A(t,r)\partial_r u_\varepsilon] dr \\ \leqslant & \ 2K[\|u_\varepsilon(T)\|_{L^1(\mathbb{R}^d_x\times\mathbb{R}_v)} + \|u_\varepsilon(0)\|_{L^1(\mathbb{R}^d_x\times\mathbb{R}_v)}] + \int_0^T dt \int_{\mathbb{R}^d} dx \int_{-K}^{K} A(t,v) u_\varepsilon(t,x,v) dv \\ & - \int_0^T dt \int_{\mathbb{R}^d} dx \int_{-K}^{K} dv \int_{-K}^{v} \partial_r A(t,r) u_\varepsilon(t,x,r) dr. \end{aligned}$$

Combining (3.12), we arrive at

$$\begin{aligned} & \int_0^T dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}} m_\varepsilon(t,x,v) dv \\ \leqslant & \ 4K^2 \|\rho_0\|_{L^1(\mathbb{R}^d)} + (1+2K) \int_0^T dt \int_{\mathbb{R}^d} dx \int_{-K}^{K} \|\partial_v A(t)\|_{W^{1,\infty}(\mathbb{R})} |u_\varepsilon(t,x,v)| dv. \end{aligned}$$

Whence m_ε is bounded uniformly in ε.

By extracting a unlabeled subsequence, one achieves

$$m_\varepsilon \to m \geqslant 0 \text{ in } \mathcal{D}'([0,\infty) \times \mathbb{R}^d_x \times \mathbb{R}_v).$$

In order to show m yields the properties stated in Lemma 1.1, it suffices to test that it is continuous in t, and by a translation, it remains to demonstrate the continuity at zero. But this fact is obvious, so the required result is complete.

• **Assertion 5:** $u(t,x,v) = \chi_{\rho(t,x)}(v)$ and ρ solves (1.1), (1.2). In addition, for any $\rho_0, \tilde{\rho}_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, the related solutions u and $\tilde{u}$ of (3.1) fulfill

$$\begin{aligned} \|[u(t)-\tilde{u}(t)]^+\|_{L^1(\mathbb{R}_x^d\times\mathbb{R}_v)} &\leqslant \exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds)\|[\chi_{\rho_0}-\chi_{\tilde{\rho}_0}]^+\|_{L^1(\mathbb{R}_x^d\times\mathbb{R}_v)} \\ &= \exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds)\|[\rho_0-\tilde{\rho}_0]^+\|_{L^1(\mathbb{R}^d)}, \end{aligned} \tag{3.20}$$

$$\|\rho(t)-\tilde{\rho}(t)\|_{L^1(\mathbb{R}^d} \leqslant \exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds)\|\rho_0-\tilde{\rho}_0\|_{L^1(\mathbb{R}^d)}, \tag{3.21}$$

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \exp(\int_0^t \|[\partial_v A]^+\|_{L^\infty(\mathbb{R})}(s)ds)\|\rho_0\|_{L^\infty(\mathbb{R}^d)}. \tag{3.22}$$

Furthermore, if $\rho_0 \leqslant \tilde{\rho}_0$, for almost all $(t,x,v) \in (0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v$, and almost all $(t,x) \in (0,\infty)\times\mathbb{R}^d$,

$$u(t,x,v) \leqslant \tilde{u}(t,x,v), \quad \rho(t,x) \leqslant \tilde{\rho}(t,x). \tag{3.23}$$

In particular, if $\rho_0 \geqslant 0$, then $u \geqslant 0$, $\rho \geqslant 0$.

Observing that $u_\varepsilon \to u$, $\rho_\varepsilon \to \rho$ and $m_\varepsilon \to m$, so $u_\varepsilon(t,x,v) - \chi_{\rho_\varepsilon}(v) \to 0$ and then $u = \chi_{\rho(t,x)}(v)$. Moreover ρ is a weak solution of (3.1).

With the help of (3.10) − (3.13), the rest of the assertion is clear.

Step 2: Existence of stochastic weak solutions to (1.7).

Before handling the general σ, we review some notions. For any $a \in \mathbb{R}^d$, set τ_a by

$$\tau_a\varphi(x) = \varphi(x+a), \quad \text{for any } \varphi \in \mathcal{C}(\mathbb{R}^d),$$

and the pullback mapping of m by τ_a^* is defined by

$$\tau_a^* m(\tilde{\phi}) = m(\tau_{-a}\tilde{\phi}) = \int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}} \tilde{\phi}(t,x-a,v)dv,$$

for any $\tilde{\phi} \in \mathcal{D}([0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v)$.

Let us consider the below Cauchy problem

$$\left\{ \begin{array}{l} \partial_t\tilde{u}(t,x,v) + b(v)\cdot\nabla_x\tilde{u} + A(t,v)\partial_v\tilde{u}(t,x,v) = \tau_{M(t)}^*\partial_v m, \quad \text{in} \;\; (0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v, \\ \tilde{u}(t,x,v)|_{t=0} = \chi_{\rho_0(x)}(v), \quad \text{in} \;\; \mathbb{R}_x^d\times\mathbb{R}_v. \end{array} \right. \tag{3.24}$$

The arguments employed in (3.1) for $\partial_v m$ adapted to $\tau_{M(t)}^*\partial_v m = \partial_v\tau_{M(t)}^* m$ in (3.24) now, produces that there is a $\tilde{u}(\omega) \in L_{loc}^\infty([0,\infty); L^\infty(\mathbb{R}_x^d\times\mathbb{R}_v)) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}_x^d\times\mathbb{R}_v))$ solving (3.24). Note that $\tau_{M(t)}^*\partial_v m$ is $\mathcal{F}_t$-adapted with values in $\mathcal{D}'(\mathbb{R}_x^d\times\mathbb{R}_v)$, thus for any for any $\phi \in \mathcal{D}(\mathbb{R}_x^d\times\mathbb{R}_v)$, $\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\tilde{u}(t,x,v)\phi(x,v)dxdv$ is $\mathcal{F}_t$-adapted. Besides, by Assertion 5, $\tilde{u} \in L^\infty(\Omega; L_{loc}^\infty([0,\infty); L^\infty(\mathbb{R}_x^d\times\mathbb{R}_v))) \cap \mathcal{C}([0,\infty); L^1(\mathbb{R}_x^d\times\mathbb{R}_v\times\Omega))$.

Hence upon using Itô-Wentzell's formula (see [8]) to

$$G(y) = \int_{\mathbb{R}_x^d\times\mathbb{R}_v}\tilde{u}(t,x,v)\phi(x+y,v)dxdv,$$

one gains

$$
\begin{aligned}
&\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\tilde{u}(t,x,v)\phi(x+M_t,v)dxdv-\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\chi_{\rho_0(x)}(v)\phi(x,v)dxdv\\
=&\int_0^t ds\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\tilde{u}b(v)\cdot\nabla_x\phi(x+M_s,v)dxdv+\int_0^t ds\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\tilde{u}\partial_v[A(s,v)\phi(x+M_s,v)]dxdv\\
&+\int_0^t M_i(\circ ds)\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\tilde{u}(s,x,v)\partial_{x_i}\phi(x+M_s,v)dxdv-\int_0^t\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\partial_v\phi(x,v)m(ds,dx,dv).
\end{aligned}
$$

Let $u(t,x,v)=\tilde{u}(t,x-M_t,v)$, then $\tilde{u}\in L^\infty(\Omega;L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}_x^d\times\mathbb{R}_v)))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}_x^d\times\mathbb{R}_v\times\Omega))$, which is $\mathcal{F}_t$-adapted, and

$$
\begin{aligned}
&\int_{\mathbb{R}_x^d\times\mathbb{R}_v}u(t,x,v)\phi(x,v)dxdv-\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\chi_{\rho_0(x)}(v)\phi(x,v)dxdv\\
=&\int_0^t ds\int_{\mathbb{R}_x^d\times\mathbb{R}_v}u(s,x,v)b(v)\cdot\nabla_x\phi(x,v)dxdv+\int_0^t ds\int_{\mathbb{R}_x^d\times\mathbb{R}_v}u(s,x,v)\partial_v[A(s,v)\phi(x,v)]dxdv\\
&+\int_0^t M_i(\circ ds)\int_{\mathbb{R}_x^d\times\mathbb{R}_v}u(s,x,v)\partial_{x_i}\phi(x,v)dxdv-\int_0^t\int_{\mathbb{R}_x^d\times\mathbb{R}_v}\partial_v\phi(x,v)m(ds,dx,dv).
\end{aligned}\tag{3.25}
$$

Thanks to (3.25) and Remark 3.1, hence there exists a stochastic weak solution to (1.7).

Step 3: Existence of stochastic entropy solutions to (1.1), (1.2).

Due to Step 2, one claims that

$$u(t,x,v)=\chi_{\rho(t,x)}(v)\ \text{and}\ \rho\in L^\infty(\Omega;L^\infty_{loc}([0,\infty);L^\infty(\mathbb{R}^d)))\cap\mathcal{C}([0,\infty);L^1(\mathbb{R}^d\times\Omega)).$$

Lemma 1.1 (ii) applies, ρ is a stochastic entropy solution of (1.1), (1.2).

With the help of Theorem 3.1 [7], and Assertion 5, we have:

Corollary 3.1 *Let ρ_0, B, σ and A meet the conditions mentioned in Theorem 3.1, then there is a unique stochastic entropy solution of* (1.1), (1.2). *Furthermore, if $\rho_0\geqslant 0$, then the unique stochastic entropy solution $\rho\geqslant 0$.*

Remark 3.2 *(i) Our proof for Theorem 3.1 is inspired by the proof for Theorem 1 in [9] and Lemma 2.1 in [10]. For more details, one can see [9-10] and the references cited up there.*

(ii) When $A(t,\rho)=\xi(t)\rho(t,x)$, then an analogue calculation of (3.21), (3.22) *also yields that*

$$\|\rho(t)\|_{L^\iota(\mathbb{R}^d)}\leqslant\exp\Big(\int_0^t\xi(s)ds\Big)\|\rho_0\|_{L^\iota(\mathbb{R}^d)},\quad \textit{for}\ \ t\in[0,\infty)\ \ \textit{and}\ \ \iota=1,\infty.$$

Whence for any $p\in[1,\infty]$,

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)}\leqslant C\exp\Big(\int_0^t\xi(s)ds\Big).\tag{3.26}$$

If there is a positive real number $c>0$ such that $\xi\leqslant -c$, then with probability one, the unique stochastic entropy solution ρ is exponentially stable. If for some real number $\alpha_1,r_1>0$, ξ possesses the below form

$$\xi(t)=\begin{cases}-\frac{\alpha_1}{t}, & \textit{when}\ \ t\in(r_1,\infty),\\ \xi_1(t), & \textit{when}\ \ t\in[0,r_1],\end{cases}$$

where $\xi_1 \in L^1([0,r_1])$, *then from* (3.26),

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leqslant \frac{C}{t^{\alpha_1}}, \quad t \in [0,\infty),$$

which implies ρ *is asymptotically stable.*

(iii) (1.7) *can be solvable by the stochastic characteristic method for general forcing* $A = A(t,x,\rho)$ *and flux* $B = B(t,x,\rho)$, *when* $b(t,x,v) = \partial_v B(t,x,v)$ *and* A *are Lipschitz. But in Theorem 3.1, this method is not used, for one may establish the fundamental existence results on weak solutions for a large scale on* b *and* A. *Then the renormalization technique applies, one may obtain the existence of stochastic weak solutions for* (1.7) *on weaker assumptions on* b *and* A, *especially for the case of non-Lipschitz vector fields (for example one can see [11] and [12] for the linear stochastic transport equation). And now if the weak solution is unique, it can be given by: for any* $\phi \in \mathcal{D}(\mathbb{R}^d_x \times \mathbb{R}_v)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d_x\times\mathbb{R}_v} u(t,x,v)\phi(x,v)dxdv \\
=\ &\int_{\mathbb{R}^d_x\times\mathbb{R}_v} \chi_{\rho_0(x)}(v)\phi(X_{0,t},V_{0,t})\exp\Big(\int_0^t \partial_v A(r,X_{0,r}V_{0,r})dr\Big)dxdv \\
&-\int_0^t\int_{\mathbb{R}^d_x\times\mathbb{R}_v} \partial_v[\phi(X_{s,t},V_{s,t})\exp\Big(\int_s^t \partial_v A(r,X_{s,r},V_{s,r})dr\Big)]m(ds,dx,dv),
\end{aligned}$$

where

$$\left\{\begin{array}{l}
\frac{d}{dt}(X_{s,t}(x,v),V_{s,t}(v))^\top = (b(t,X_{s,t},V_{s,t}) + \dot{M}(t), A(t,X_{s,t},V_{s,t}))^\top, \ t \geqslant s, \\
(X_{s,t}(x,v),V_{s,t}(v))^\top|_{t=s} = (X(s,x,v),V(s,x,v))^\top,
\end{array}\right.$$

and $M(t) = (M_1(t), M_2(t), \ldots, M_d(t))$.

(iv) All above restrictions on B *and* A *seem to be rigid, but there are many models in statistic physics and fluid mechanics, satisfying all the hypotheses, we intend to illustrate it by displaying a typical example now.*

Example 3.1 *This example is concerned with the Buckley-Leverett equation (see [13]), which provides a simple model for the rectilinear flow of immiscible fluids (phases) through a porous medium. To be simple, nevertheless, to capture some of the qualitative features, here we consider the case of two-phase flows (such as oil and water) in one space dimension. In this issue, the Buckley-Leverett equation, with an external force, and a stochastic perturbation reads*

$$\left\{\begin{array}{l}
\partial_t\rho(t,x) + \partial_x B(\rho) + \partial_x\rho(t,x)\circ\dot{M}(t) = \mu A(t,\rho), \quad in \ \ \Omega\times(0,\infty)\times\mathbb{R}, \\
\rho(t,x)|_{t=0} = \rho_0(x), \quad in \ \ \mathbb{R},
\end{array}\right. \tag{3.27}$$

where

$$B(\rho) = \left\{\begin{array}{cl}
0, & when \ \ \rho < 0, \\
\frac{\rho^2}{\rho^2+(1-\rho)^2}, & when \ \ 0 \leqslant \rho \leqslant 1, \\
1, & when \ \ \rho > 1,
\end{array}\right. \quad M(t) = \int_0^t \vartheta(t)dW(s), \quad A(t,\rho) = \frac{\theta(t)\rho^2}{1+\rho^2},$$

$\mu \geqslant 0$ *is a constant,* W *is a 1-dimensional standard Wiener process,* $\theta \in L^1_{loc}([0,\infty))$. *Then* $B \in \mathcal{C}^1$ *and* $A \in L^1_{loc}([0,\infty); W^{1,\infty}(\mathbb{R}))$, $A(0) = 0$.

Applying the Corollary 3.1, we obtain

Corollary 3.2 *Assume that $\vartheta \in L^2_{loc}([0,\infty))$, $\rho_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a unique stochastic entropy solution ρ of (3.27). Moreover, if $\rho_0 \geqslant 0$, then $\rho \geqslant 0$.*

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